Comparative study of nonclassicality, entanglement, and dimensionality of multimode noisy twin beams

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The nonclassicality, entanglement, and dimensionality of a noisy twin beam are determined using a characteristic function of the beam written in the Fock basis. One-to-one correspondence between the negativity quantifying entanglement and the nonclassicality depth is revealed. Twin beams, which are either entangled or nonclassical (independent of their entanglement), are observed only for the limited degrees of noise, which degrades their quantumness. The dimensionality of the twin beam quantified by the participation ratio is compared with the dimensionality of entanglement determined from the negativity. Partitioning of the degrees of freedom of the twin beam into those related to entanglement and to noise is suggested. Both single-mode and multimode twin beams are analyzed. Weak nonclassicality based on integrated-intensity quasidistributions of multimode twin beams is studied. The relation of the model to the experimental twin beams is discussed.

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I. INTRODUCTION

The question whether a given state cannot be described within a classical theory has been considered one of the most serious since the early days of quantum physics [1–3] (for a review see, e.g., Ref. [4]). Nonclassicality and entanglement, which is one of the nonclassicality manifestations, are the most important properties of optical fields studied in quantum optics. Such fields have no classical analogs and as such they have been found interesting for many reasons. Nonclassical properties of such fields have been found useful both for elucidating the principles of quantum mechanics and in various applications including, e.g., quantum information processing [5], quantum metrology [6–8], and highly-sensitive measurements [9].

From both the theoretical and the experimental points of view, the nonlinear process of parametric down-conversion, in which photon pairs are generated, has played an important role here from the beginning of investigations [4, 10–12]. Its individual photon pairs have been exploited in many fundamental experiments testing nonclassical behavior predicted by quantum physics [13, 14]. It has also allowed the generation of more intense fields having their electric-field amplitude quadratures squeezed below the vacuum level [15–17], exhibiting sub-shot-noise correlations [18, 19] or having sub-Poissonian photon-number statistics [20–22].

In quantum optics, the definition of nonclassicality is based upon the Glauber-Sudarshan P representation [11, 23, 24] of the statistical operator of a given field. The commonly accepted formal criterion for distinguishing nonclassical states from classical ones is expressed as follows [10, 11, 25, 26]: A quantum state is *nonclassical* if its Glauber-Sudarshan P function fails to have the properties of a probability density. Alternatively, several operational criteria for nonclassicality of either single-mode [25–27] or multimode [28–30] fields have been revealed. Their derivations are based either on field's moments [28, 30, 31] or on direct reconstruction of quasidistributions of integrated intensities [32–34]. Also, criteria derived from the majorization theory have been found [35, 36].

Entanglement (or inseparability) is a special nonclassical property that describes quantum correlations among (in general) several subsystems that cannot be treated by the means of classical statistical theory [37]. Various approaches have been developed for discrete and continuous variables to reveal entanglement. This property has been exploited in suggesting an entanglement criterion and the related entanglement measure (referred to as the negativity) based upon the partial transposition of a statistical operator [38–41]. Another approach has been based on the violation of the Bell inequalities written for different mean values including the measurement on both parts of a bipartite system [42]. Also, a method using positive semi-definite matrices of fields' moments of different orders [43, 44] has been found to be very powerful. We would like to stress at this point that entanglement is a very crucial tool in today's quantum information processing.

In this contribution, we study nonclassicality by applying the Lee nonclassical depth [45] as well as entanglement via the negativity [40, 41] for (in general) noisy twin beams of different intensities. Such fields occur under real experimental conditions in which a nonlinear crystal generates both photon pairs and individual single photons

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(noise). Nevertheless, the signal and idler fields together form a bipartite quantum system. We note that entanglement and nonclassicality of twin beams generated by down-conversion seeded by thermal light have been analyzed in Refs. [46–48]. In this case, noise present in the incident thermal fields participates in the nonlinear process and generation of photon pairs. This weakens its detrimental effect on entanglement and nonclassicality of twin beams and allows us to have entangled twin beams with a larger amount of noise.

Here we also study the problem of entanglement dimension via the negativity N for general twin beams and the Schmidt number K for noiseless twin beams in a pure state. Namely, we estimate how many degrees of freedom of two fields comprising a twin beam are entangled based on the results of Ref. [49] for axisymmetric states. On the other hand, the participation ratio $R_{\rm s}$ [50] determined from the reduced statistical operator $\hat\rho_{\rm s}$ of the signal (or idler) field gives the number of degrees of freedom in this field serving to describe both entanglement and noise. It varies from $R_{\rm s}=1$ (for a pure state $\hat\rho_{\rm s}$) to $R_{\rm s}=d=\dim(\hat\rho_{\rm s})$ for the completely mixed state $\hat\rho_{\rm s}=I/d$. We note that the participation ratio $R_{\rm s}$ gives an effective number of states in the mixture $\hat\rho_{\rm s}$ implied by the property that it is a lower bound for the rank of $\hat\rho_{\rm s}$. Moreover, the logarithm of R is the von Neumann–Renyi entropy of second order [50]. The inverse of the participation ratio is referred to as the purity (or linear entropy). Various methods for direct measuring the Schmidt number K (even without recourse to quantum tomography) were proposed for noiseless twin beams (see, e.g., Refs. [51–55]). The method of Ref. [53] was recently realized experimentally [56]. We note that the negativity can also be measured without applying quantum tomography as described, e.g., for two polarization qubits using linear optical setups [57, 58].

The paper is organized as follows. In Sec. II, the model of parametric down-conversion providing an appropriate statistical operator of a twin beam is presented. Entanglement of the twin beam is addressed in Sec. III using the negativity. The nonclassical depth is introduced in Sec. IV to quantify nonclassicality. The relation between the negativity and the nonclassical depth is also discussed in Sec. IV. The dimensionality of a twin beam described by the participation ratio together with the entanglement dimensionality described by the negativity is analyzed in Sec. V. Properties of M-mode twin beams are discussed in Sec. VI. Section VII is devoted to experimental multimode twin beams containing also noise embedded in independent spatiotemporal modes. Conclusions are drawn in Sec. VIII.

II. QUANTUM MODEL OF A TWIN BEAM

To describe the generation of a single-mode twin beam by parametric down-conversion, we adopt the approach based on the Heisenberg equations derived from the appropriate nonlinear Hamiltonian $\hat H_{\rm int}$ [11],

$$\hat H_{\rm int} = -\hbar g \hat A_1 \hat A_2 \exp({\rm i}\omega t - {\rm i}\phi) + {\rm H.c.}, \qquad (1)$$

where $\hat A_1$ ($\hat A_1^\dagger$) and $\hat A_2$ ($\hat A_2^\dagger$) represent the annihilation (creation) operators of the signal and idler field, respectively, and g is a real coupling constant that is linearly proportional both to the quadratic susceptibility of a nonlinear medium and to the real pump-field amplitude. The interaction time is denoted t, ω (ϕ) is the pump-field frequency (phase), and ω_1 and ω_2 stand for the signal- and idler-field frequencies, respectively. The law of energy conservation provides the relation $\omega=\omega_1+\omega_2$. H.c. is the Hermitian conjugated term. In a real nonlinear process, also noise occurs. It can be described by the Langevin forces $\hat L$ belonging to a reservoir of chaotic oscillators with mean number of noise photons $\langle n_{\rm d}\rangle$.

The Heisenberg-Langevin equations corresponding to the Hamiltonian $\hat H_{\rm int}$ are written as

$$\begin{aligned} \frac{{\rm d}\hat A_1}{{\rm d}t} &= -({\rm i}\omega_1+\gamma_1)\hat A_1 + {\rm i}g\hat A_2^\dagger \exp(-{\rm i}\omega t+{\rm i}\phi) + \hat L_1, \\ \frac{{\rm d}\hat A_2}{{\rm d}t} &= -({\rm i}\omega_2+\gamma_2)\hat A_2 + {\rm i}g\hat A_1^\dagger \exp(-{\rm i}\omega t+{\rm i}\phi) + \hat L_2, \end{aligned} \qquad (2)$$

where the constant γ_1 (γ_2) describes damping in the signal (idler) field. The Langevin operators $\hat L_i$ (for $i=1,2$) have the properties

$$\begin{aligned} \langle \hat L_i\rangle = \langle \hat L_i^\dagger\rangle = 0, \langle \hat L_i^\dagger \hat L_j\rangle = 2\gamma_j\langle n_{\rm d}\rangle\delta_{ij}, \\ \langle \hat L_i \hat L_j^\dagger\rangle = 2\gamma_j\left(\langle n_{\rm d}\rangle+1\right)\delta_{ij}, \end{aligned} \qquad (3)$$

where δ_{ij} stands for the Kronecker symbol.

Using the interaction representation [$\hat A_j(t) = a_j(t)\exp(-i\omega_j t)$] and neglecting damping together with the Langevin forces, the solution of Eq. (2) attains the form

$$\begin{aligned} \hat a_1(t) &= \hat a_1(0)u(t) + i\hat a_2^\dagger(0)v(t)\exp(i\phi), \\ \hat a_2(t) &= \hat a_2(0)u(t) + i\hat a_1^\dagger(0)v(t)\exp(i\phi), \end{aligned} \qquad (4)$$

in which $u(t)=\cosh(gt)$ and $v(t)=\sinh(gt)$.

Statistical properties of the twin beam are then described by the normal characteristic function $C_{\mathcal N}$ defined as

$$C_{\mathcal N}(\beta_1,\beta_2) = {\rm Tr}\left[\hat\rho\exp(\beta_1\hat a_1^\dagger+\beta_2\hat a_2^\dagger)\exp(-\beta_1^*\hat a_1-\beta_2^*\hat a_2)\right], \qquad (5)$$

where Tr denotes the trace. Using the solution given in Eq. (4), the normal characteristic function $C_{\mathcal N}$ attains the Gaussian form [59],

$$\begin{aligned} C_{\mathcal N}(\beta_1,\beta_2) = \exp\big[&-(|\beta_1|^2B_1+|\beta_2|^2B_2) + D_{12}\beta_1^*\beta_2^* + \\ &+D_{12}^*\beta_1\beta_2\big], \end{aligned} \qquad (6)$$

in which β_1 and β_2 denote independent complex variables. For the undamped and noiseless case, we have

$D_{12} = \langle \triangle \hat{a}_1 \triangle \hat{a}_2 \rangle$. Also the mean number $B_{\rm p}$ of the generated photon pairs is determined as $B_{\rm p} = \langle \triangle \hat{a}_1^\dagger \triangle \hat{a}_1 \rangle = \langle \triangle \hat{a}_2^\dagger \triangle \hat{a}_2 \rangle$. When damping and noise are also considered [59], the parameters B_a (for $a = 1, 2$) contain additional noise contributions characterized by the parameters $B_{\rm s}$ and $B_{\rm i}$, i.e., $B_1 = B_{\rm p} + B_{\rm s}$ and $B_2 = B_{\rm p} + B_{\rm i}$. Whereas the parameter $B_{\rm p}$ gives the mean number of photon pairs, the parameters $B_{\rm s}$ and $B_{\rm i}$ correspond to the mean number of noise photons coming from the signal- and idler-field reservoirs, respectively. On the other hand, the parameter D_{12} describing mutual correlations between the signal and idler fields is not influenced by the noise since $|D_{12}|^2 = B_{\rm p}(B_{\rm p} + 1)$.

The statistical operator $\hat{\rho}$ of the twin beam then acquires the form [11]

$$\hat{\rho} = \frac{1}{\pi^2} \int d^2\beta_1 d^2\beta_2 \mathcal{C}_{\mathcal{A}}(\beta_1, \beta_2) : \exp\left(\sum_{j=1}^{2} \hat{a}_j \beta_j^* - \hat{a}_j^\dagger \beta_j \right) : . \tag{7}$$

In Eq. (7), $C_{\mathcal{A}}(\beta_1, \beta_2) = C_{\mathcal{N}}(\beta_1, \beta_2) \exp(-|\beta_1|^2 - |\beta_2|^2)$ denotes an anti-normal characteristic function and symbol : : means normal ordering of field operators.

Performing integration in Eq. (7) we express the statistical operator $\hat{\rho}$ in the form

$$\hat{\rho} = \frac{1}{\tilde{K}} : \exp\left[-\frac{\tilde{B}_2}{\tilde{K}} \hat{a}_1^\dagger \hat{a}_1 - \frac{\tilde{B}_1}{\tilde{K}} \hat{a}_2^\dagger \hat{a}_2 + \frac{|D_{12}|}{\tilde{K}} \left(\hat{a}_1 \hat{a}_2 + \hat{a}_1^\dagger \hat{a}_2^\dagger \right) \right] :, \tag{8}$$

where $\tilde{K} = \tilde{B}_1 \tilde{B}_2 - |D_{12}|^2$. The parameters $\tilde{B}_a$ introduced in Eq. (8) are related to anti-normal ordering of field operators and are given as $\tilde{B}_a = B_a + 1$ with $a = 1, 2$. Decomposing the statistical operator $\hat{\rho}$ in the Fock-state basis we finally arrive at the formula

$$\rho_{ij,kl} = \sum_{n=0}^{\infty} \sum_{p=0}^{n} \sum_{r=0}^{p} \sum_{t=0}^{r} (-1)^{n-r} \frac{\tilde{B}_2^{n-p} \tilde{B}_1^{p-r} \tilde{K}^{-n-1}}{(n-p)!(p-r)!} \times \frac{|D_{12}|^r}{(r-t)!\,t!} \langle ij | \hat{a}_1^{\dagger n-p+t} \hat{a}_2^{\dagger p-r+t} \hat{a}_1^{n-p+r-t} \hat{a}_2^{p-t} | kl \rangle. \tag{9}$$

Direct inspection of Eq. (9) for the matrix elements of the statistical operator $\hat{\rho}$ written in Eq. (9) reveals that all nonzero elements can be parameterized by only three indices,

$$\rho_{i,j,i+d,j+d} = \frac{1}{\tilde{K}} \sqrt{\frac{(i+d)!}{i!} \frac{(j+d)!}{j!}} \sum_{m=0}^{\max(i,j)} C_m^i C_m^j \times \frac{m!}{(m+d)!} X_1^{j-m} X_2^{i-m} \left(\frac{|D_{12}|}{\tilde{K}} \right)^{d+2m}, \tag{10}$$

assuming $d \geq 0$. Moreover, $\rho_{ij,i+d,j+d} = \rho_{i+d,j+d,i,j}$, $X_a = 1 - \tilde{B}_a / \tilde{K}$ with $a = 1, 2$, and C_m^i and C_m^j denote the binomial coefficients.

III. NEGATIVITY OF THE TWIN BEAM

The negativity N of a mixed bipartite system defined on the basis of the Peres-Horodecki criterion for a partially transposed statistical operator [38, 39, 41] is useful for quantifying the entanglement of the twin beam. It can be expressed as

$$N(\hat{\rho}) = \frac{||\hat{\rho}^\Gamma||_1 - 1}{2} \tag{11}$$

using the trace norm $||\rho^\Gamma||_1$ of the partially transposed statistical operator ρ^Γ. The negativity essentially measures the degree at which ρ^Γ fails to be positive. As such it can be regarded as a quantitative version of the Peres-Horodecki criterion for separability [38, 39]. According to Eq. (11), the negativity N is given as the absolute value of the sum of the negative eigenvalues of ρ^Γ. It vanishes for separable states. It is worth noting that the negativity N is an entanglement monotone and so it can be used to quantify the degree of entanglement in bipartite systems. Moreover, the negativity does not reveal bound entanglement (i.e., nondistillable entanglement) in systems more complicated than two qubits or qubit-qutrit [37].

To determine the negativity N we consider the eigenvalue problem for the partially transposed statistical operator $\hat{\rho}^\Gamma$. The statistical operator $\hat{\rho}^\Gamma$ expressed in the Fock-state basis attains a characteristic block structure. The smallest block has dimension 2 and each successive block has dimension larger by 1. For a given M one has a block of dimension $M + 1$. Such a block represents a matrix of $M + 1$ isolated states; the sum of indices of their statistical operators equals $2M$,

$$\hat{\rho}_M^\Gamma = \begin{pmatrix} \rho_{0\,M,0\,M} & \rho_{0\,M-1,1\,M} & \cdots & \rho_{0\,0,M\,M} \\ \rho_{1\,M,0\,M-1} & \rho_{1\,M-1,1\,M-1} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \rho_{M\,M,0\,0} & \cdots & \cdots & \rho_{M\,0,M\,0} \end{pmatrix}. \tag{12}$$

It can be shown that eigenvalues of a block of dimension $M + 1$ can be expressed as $\nu_+^M, \nu_+^{M-1}\nu_-, \ldots, \nu_+\nu_-^{M-1}, \nu_-^M$ using the eigenvalues ν_+ and ν_- of a block with dimension 2:

$$\nu_\pm = 1 - \frac{1}{2\tilde{K}} \left(\tilde{B}_1 + \tilde{B}_2 \mp \sqrt{\left(\tilde{B}_2 - \tilde{B}_1 \right)^2 + 4|D_{12}|^2} \right). \tag{13}$$

The negative eigenvalues can only be those containing odd powers of ν_-. They form a geometric progression whose elements can be summed to arrive at the formula for the negativity N:

$$N = \frac{1}{2} \frac{3(\tilde{B}_1 + \tilde{B}_2) + \sqrt{(\tilde{B}_1 - \tilde{B}_2)^2 + 4|D_{12}|^2} - 4\tilde{K} - 2}{4\tilde{K} - 2(\tilde{B}_1 + \tilde{B}_2) + 1}. \tag{14}$$

Expressing parameters $\tilde{B}_1$, $\tilde{B}_2$, and $|D_{12}|^2$ in Eq. (14) in terms of parameters $B_{\rm p}$, $B_{\rm s}$, and $B_{\rm i}$, we arrive at the

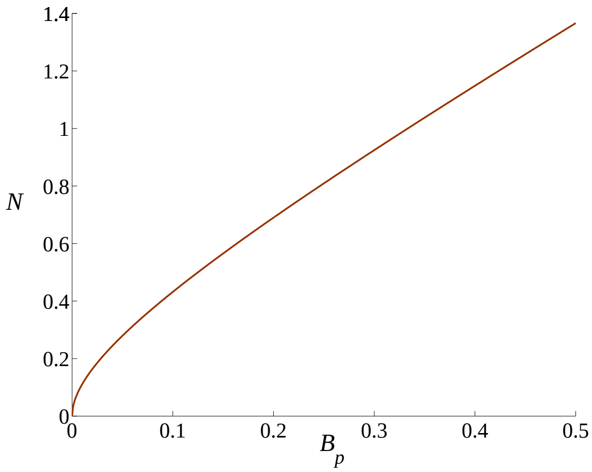

FIG. 1: (Color online) Negativity N as a function of the mean photon-pair number $B_{\rm p}$ for noiseless twin beams (i.e., $B_{\rm s} = B_{\rm i} = 0$)) according to Eq. (16).

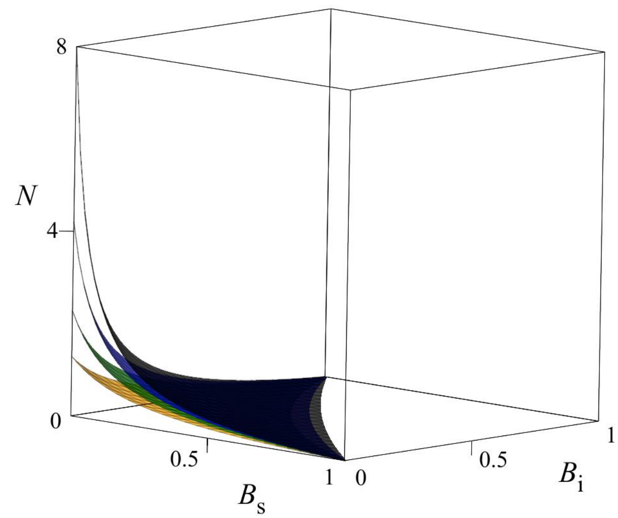

FIG. 2: (Color online) Negativity N, given in Eq. (15), as a function of the mean noise photon numbers $B_{\rm s}$ and $B_{\rm i}$ in the signal and idler modes, respectively, assuming the mean photon-pair number $B_{\rm p}$ equal to 0.5 [bottom light-gray (yellow) area], 1 [gray (green) area], 2 [dark-gray (blue) area] and 4 [top, black area]. The larger $B_{\rm p}$, the larger the negativity N.

formula

$$\begin{aligned} N &= \big\{2B_{\rm p} - (B_{\rm s}+B_{\rm i})(4B_{\rm p}+1) - 4B_{\rm s}B_{\rm i} \\ &\quad + \sqrt{(B_{\rm s}-B_{\rm i})^2 + 4B_{\rm p}(B_{\rm p}+1)}\big\} \\ &\quad \times \big\{4(B_{\rm s}+B_{\rm i})(2B_{\rm p}+1) + 8B_{\rm s}B_{\rm i} + 2\big\}^{-1}. \end{aligned} \tag{15}$$

Equation (15) simplifies considerably for noiseless twin beams:

$$N = B_{\rm p} + \sqrt{B_{\rm p}(B_{\rm p}+1)}. \tag{16}$$

According to Eq. (16), all noiseless twin beams are entangled. The more intense the noiseless twin beams are, the more entangled the signal and idler fields are (see Fig. 1). The presence of noise in a twin beam can even completely destroy entanglement, as the analysis of Eq. (15) shows. Indeed, the condition $N > 0$ for entanglement can be rewritten using Eq. (15) as follows:

$$B_{\rm p}[1-(B_{\rm s}+B_{\rm i})] > B_{\rm s}B_{\rm i}. \tag{17}$$

Condition (17) cannot be fulfilled for any value of $B_{\rm p}$ provided that $B_{\rm s}+B_{\rm i} \geq 1$. Thus, the twin beam can be entangled only when

$$B_{\rm s}+B_{\rm i} < 1 \quad \text{and} \quad B_{\rm p} > \frac{B_{\rm s}B_{\rm i}}{1-(B_{\rm s}+B_{\rm i})}. \tag{18}$$

The behavior of the negativity N of noisy twin beams dependent on the noise parameters $B_{\rm s}$ and $B_{\rm i}$ is illustrated in Fig. 2 for several values of the mean photon-pair number $B_{\rm p}$. It holds in general that the greater the value of the mean photon-pair number $B_{\rm p}$, the greater the value of the negativity N. This can be explained as follows. The more intense twin beams, with their thermal statistics, are effectively spread over a larger number of the Fock states. This naturally results in the larger effectively populated Hilbert spaces used to describe the entanglement. The greater value of the negativity N means a greater effective number of the paired modes building the entanglement, i.e., a greater value of the entanglement dimensionality, as defined in Sec. V. Also, the greater the value of the mean photon-pair number $B_{\rm p}$, the larger the amount of overall noise $B_{\rm s}+B_{\rm i}$ acceptable in an entangled twin beam (see Fig. 3). The curves plotted in Fig. 3 indicate that entanglement is more resistent to noise when the noise is distributed in the signal and idler fields asymmetrically. We note that separable states (i.e., with $N=0$) contain, in general, paired, signal, and idler noisy contributions. However, the noisy contributions are sufficiently strong to suppress the "entangling power" of the photon-pair contribution and so the state effectively behaves as a classical statistical mixture of the signal and idler fields.

The decomposition of the partially transposed statistical operator $\hat{\rho}^{\Gamma}$ into blocks in its matrix representation and the fact that a block (subspace) with dimension $M+1$ describes only states with up to M photons in the signal (and also idler) field can be used to define the distribution d_N of the negativity N fulfilling the normalization condition

$$\sum_{M=1}^{\infty} d_N(M) = N. \tag{19}$$

For a given M, the element $d_N(M)$ of this distribution

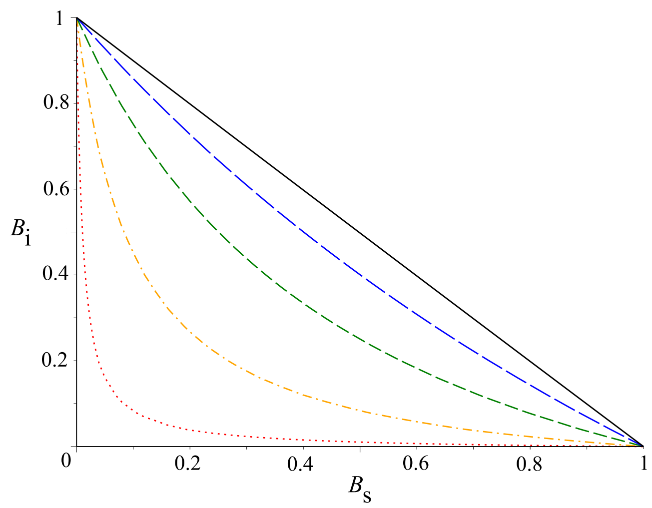

FIG. 3: (Color online) Curves giving the boundaries between entangled and separable twin beams and determined according to Eq. (18) plotted in the plane spanned by the mean noise photon numbers B_{s} and B_{i} assuming the mean photon-pair number B_{p} equal to 0.01 [dotted (red) curve], 0.1 [dash-dotted (yellow) curve], 0.5 [dashed (green) curve], 2 [long-dashed (blue) curve], and $B_{\mathrm{p}} = 100$ [solid black curve]. Entangled states are localized in the lower-left corner of the plane. The larger B_{p}, the larger the area containing entangled states.

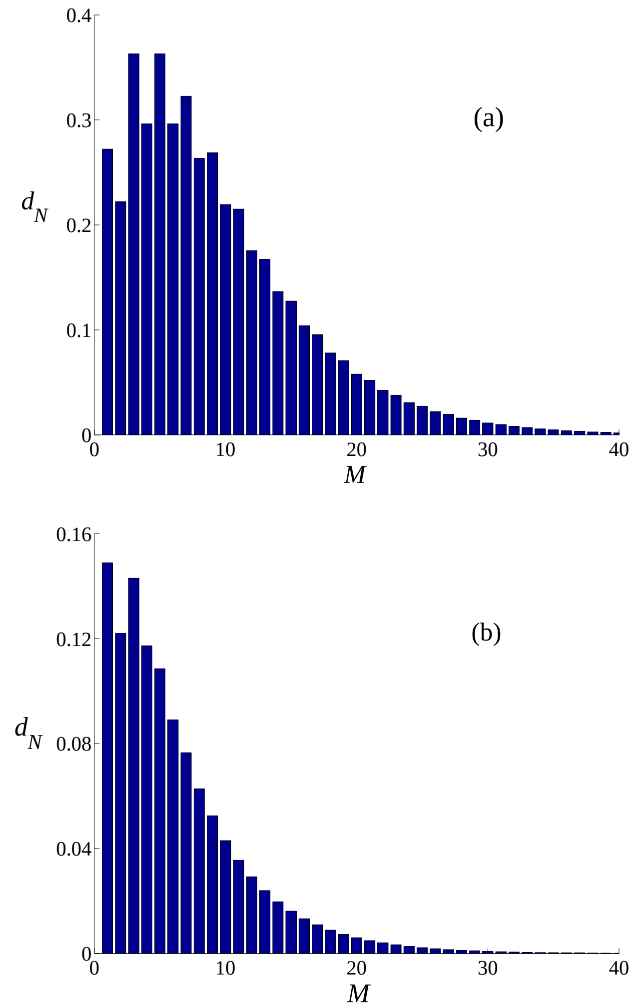

FIG. 4: (Color online) Distribution d_N of negativity N given in Eq. (19) assuming $B_{\mathrm{p}} = 2$ and (a) $B_{\mathrm{s}} = B_{\mathrm{i}} = 0$ and (b) $B_{\mathrm{s}} = B_{\mathrm{i}} = 0.1$. Note that $-d_N(M)$ corresponds to the sum of all the negative eigenvalues for the $(M+1)$–dimensional block of the partially transposed statistical operator $\hat{\rho}^{\Gamma}$. Thus, $d_N(M)$ shows the internal structure of entanglement in the Liouville space.

is given as the sum of the absolute values of the negative eigenvalues belonging to the block of dimension $M + 1$. The distribution d_N of the negativity provides insight into the internal structure of entanglement. It tells us how entanglement is distributed in the Liouville space of statistical operators. Typical distributions d_N of the negativity for noiseless as well as noisy twin beams are plotted in Fig. 4. A teeth-like structure occurs for smaller numbers M in noiseless twin beams. Noise tends to suppress this structure, as is evident from the comparison of the distributions d_N plotted in Figs. 4(a) and 4(b). We note that the densities of the negativity have already been introduced for bipartite entangled states composed of a qubit and continuum of states [60, 61] as well as two continua of states.

IV. NONCLASSICAL DEPTH OF THE TWIN BEAM

To quantify nonclassicality of the twin beam we apply the nonclassical depth τ [45] derived from the threshold value s_{th} of the ordering parameter at which the joint signal-idler quasidistribution of integrated intensities becomes nonnegative [34, 59]. We adopt the definition $\tau = (1 - s_{\mathrm{th}})/2$. We note that the joint signal-idler quasidistribution of integrated intensities attains negative values for $1 \geq s > s_{\mathrm{th}}$ for which $\tau > 0$. The threshold value s_{th} can easily be obtained from the condition $\langle [\Delta(W_{\mathrm{s}} - W_{\mathrm{i}})]^2 \rangle = 0$, which determines the point of the transition between quantum and classical single-mode twin beams [34]. This results in the following formula for the nonclassical depth τ:

$$\tau = \frac{1}{2}\left[\sqrt{(B_{\mathrm{s}} - B_{\mathrm{i}})^2 + 4B_{\mathrm{p}}(B_{\mathrm{p}} + 1)} - 2B_{\mathrm{p}} - B_{\mathrm{s}} - B_{\mathrm{i}}\right]. \tag{20}$$

Assuming noiseless twin beams, Eq. (20) simplifies to

$$\tau = \sqrt{B_{\mathrm{p}}(B_{\mathrm{p}} + 1)} - B_{\mathrm{p}}. \tag{21}$$

According to Eq. (21), all noiseless twin beams are nonclassical. The greater the mean photon-pair number B_{p}, the greater the value of the nonclassical depth τ (see Fig. 5). This depth τ reaches its greatest value, 1/2, in

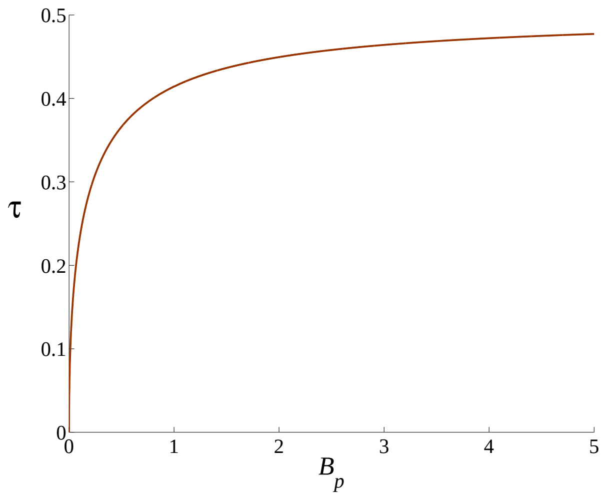

FIG. 5: (Color online) Nonclassical depth τ given in Eq. (21) as it depends on the mean photon-pair number $B_{\rm p}$ for noiseless twin beams, i.e., $B_{\rm s} = B_{\rm i} = 0$.

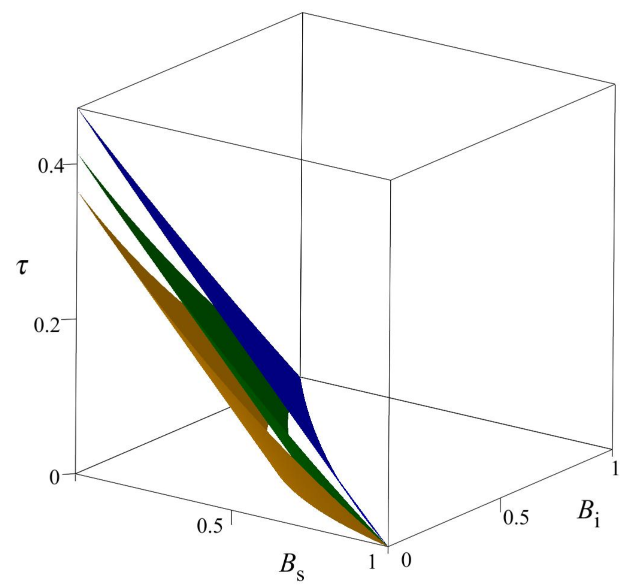

FIG. 6: (Color online) Nonclassical depth τ given in Eq. (20) as a function of the mean noise photon numbers $B_{\rm s}$ and $B_{\rm i}$ for the mean photon-pair number $B_{\rm p}$ equal to 0.1 [bottom, light-gray (yellow) area], 0.5 [gray (green) area], 4 [top, dark-gray (blue) area]. The greater the value of $B_{\rm p}$, the greater the value of τ.

the limit of an infinitely intense twin beam ($B_{\rm p} \to \infty$). We note that $\tau = 1/2$ corresponds to symmetrical ordering of the field operators.

On the other hand, and according to Eq. (20), noise only degrades nonclassical behavior of a twin beam, as documented in Fig. 6. If the noise is equally distributed in the signal and idler fields ($B_{\rm s} = B_{\rm i}$), the nonclassical depth τ determined along Eq. (20) gives the mean number $B_{\rm s} + B_{\rm i}$ of noise photons needed for suppressing the nonclassicality of the twin beam. So, the larger the value of the nonclassical depth τ is, the more nonclassical the field is. On the other hand, formal application of Eq. (20) to classical noisy twin beams results in negative values of the nonclassical depth τ. Their absolute value $|\tau|$ can be considered a measure of classicality of noisy twin beams in the sense that it quantifies the mean number of photon pairs needed to transform a classical twin beam into the classical-quantum boundary $\tau = 0$.

Condition $\tau = 0$ for the transition from quantum to classical twin beams applied to Eq. (20) results in the same relation among parameters $B_{\rm p}$, $B_{\rm s}$, and $B_{\rm i}$ as derived in Eq. (17) for the boundary between entangled and separable twin beams. Thus, entangled twin beams are nonclassical, whereas separable twin beams are classical. This means that nonclassical twin beams may contain on average only less than one noise photon ($B_{\rm s} + B_{\rm i} < 1$). We note that inequality (17) represents the Simon criterion for nonclassicality of Gaussian states as shown in Ref. [62].

Comparison of Eqs. (16) and (21) made for noiseless twin beams reveals a simple relation between the negativity N and the nonclassical depth τ:

$$N = \frac{\tau}{1 - 2\tau}. \tag{22}$$

Direct calculation based on Eqs. (15) and (20) then confirms that relation (22) holds even for a general noisy twin beam. We thus have a one-to-one correspondence between the value of the negativity N and the value of the nonclassical depth τ. Moreover, according to Eq. (22) the negativity N is an increasing function of the nonclassical depth τ, and vice versa (see Fig. 7). There exists a deep physical reason for this correspondence. The nonlinear process emits photons in pairs into the signal and idler fields, which creates entanglement between these fields. It is this entanglement that gives rise to nonclassical properties of twin beams, as the classical statistical optics is unable to describe pairing of photons appropriately.

V. DIMENSIONALITY OF THE TWIN BEAM

Three different numbers are needed to determine the dimensionality of a general noisy twin beam. The dimensionality $K_{\rm ent}$ of entanglement gives the number of degrees of freedom constituting the entangled (paired) part of the twin beam. We also need additional degrees of freedom to characterize the noisy parts of the twin beam. As the amount of noise is, in general, different in the signal and idler fields, we have independent participation ratios $R_{\rm s}$ and $R_{\rm i}$ for both fields. The entanglement dimensionality $K_{\rm ent}$ for bipartite states with axisymmetric statistical operators can be given in terms of the nega-

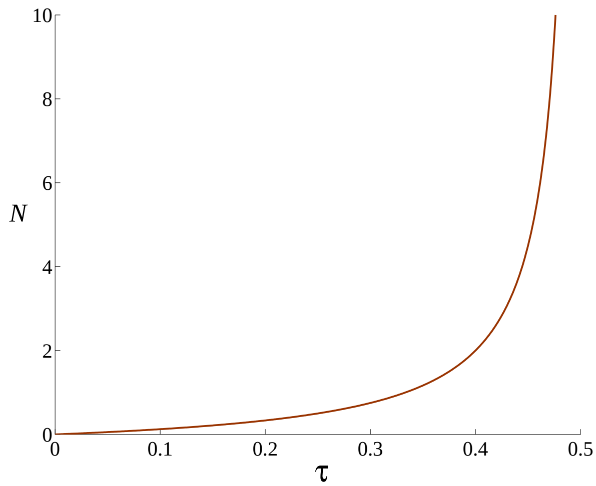

FIG. 7: (Color online) Negativity N as a function of the nonclassical depth τ, according to Eq. (22).

tivity N by a simple formula [49]:

$$K_{\rm ent}(\hat{\rho}) = 2N(\hat{\rho}) + 1 = ||\hat{\rho}^{\Gamma}||_1. \qquad (23)$$

Strictly speaking, it is the least integer $\geq K_{\rm ent}$ that gives a lower bound to the number of entangled dimensions between entangled subsystems (paired modes) of $\hat{\rho}$ [49]. According to Eq. (23), the entanglement dimensionality $K_{\rm ent}$ equals 1 for separable states ($N = 0$). It linearly increases with the negativity N. As the noise described by the mean noise photon numbers $B_{\rm s}$ and $B_{\rm i}$ decreases the values of the negativity N, it also decreases the values of the entanglement dimensionality $K_{\rm ent}$. We note that, for pure states, the Schmidt number is also a good quantifier of the entanglement dimension $K_{\rm ent}$ [63–65]. The Schmidt decomposition of pure states accompanied by convex optimization can even be applied for quantifying the entanglement dimension of mixed entangled states [37].

On the other hand, the noise present in the signal and idler fields requires additional degrees of freedom for its description. These degrees of freedom are, together with those reserved for describing entanglement, determined by the participation ratios $R_{\rm s}$ and $R_{\rm i}$ derived from the signal- and idler-field reduced statistical operators $\hat{\rho}_{\rm s}$ and $\hat{\rho}_{\rm i}$, respectively [64, 66]:

$$R_a = \frac{1}{{\rm Tr}_a[\hat{\rho}_a^2]}, \qquad a = {\rm s}, {\rm i}. \qquad (24)$$

Equation (10), giving the matrix elements of the statistical operator $\hat{\rho}$, guarantees a diagonal form of the reduced statistical operators $\hat{\rho}_{\rm s}$ and $\hat{\rho}_{\rm i}$ of the signal and idler fields, respectively. In this case, Eq. (24) can be rewritten in the form

$$R_{\rm s} = \frac{1}{\sum_j \rho_{{\rm s},jj}^2}. \qquad (25)$$

Using Eq. (10) the matrix elements $\rho_{{\rm s},jj}$ can be written as

$$\rho_{{\rm s},jj} = \frac{1}{\tilde{B}_{\rm s}} \left[\left(1 - \frac{\tilde{B}_{\rm i}}{\tilde{K}} \right) + \frac{|D_{12}|^2}{\tilde{K}\tilde{B}_{\rm s}} \right]^j. \qquad (26)$$

Substituting Eq. (26) into Eq. (25) we obtain a simple formula for the participation ratio $R_{\rm s}$:

$$R_{\rm s} = 2(B_{\rm p} + B_{\rm s}) + 1. \qquad (27)$$

The same considerations made for the signal field apply also to the idler field.

To find the relation between the entanglement dimensionality $K_{\rm ent}$ and the participation ratios $R_{\rm s}$ and $R_{\rm i}$ we consider for a while the noiseless twin beams in pure states. In this case, the elements $\hat{\rho}_{{\rm s},jj}$ of the reduced statistical operator $\hat{\rho}_{\rm s}$, written in Eq. (26), immediately give the squared Schmidt coefficients [54]. Combining Eqs. (16), (23), and (27) we arrive at the formula

$$K_{\rm ent} = R_{\rm s} + \sqrt{R_{\rm s}^2 - 1}. \qquad (28)$$

Equation (28) shows that, excluding weak noiseless twin beams, $K_{\rm ent} \approx 2R_{\rm s}$. This means that the definitions of the entanglement dimensionality and participation ratio set different boundaries for the Schmidt coefficients c_j included in the approximative description of a noiseless twin beam with the wave function

$$|\psi\rangle = \sum_{j=0}^{j_{\rm max}} c_j |j\rangle_{\rm s} |j\rangle_{\rm i}. \qquad (29)$$

Using Eq. (10), the coefficients c_j in Eq. (29) are obtained in the form

$$c_j = \sqrt{\frac{{B_{\rm p}}^j}{(B_{\rm p}+1)^{j+1}}}, \qquad (30)$$

which is in agreement with the thermal photon-number statistics of the signal (or idler) field. We note that the ratio $c_{K_{\rm ent}-1}/c_{R_{\rm s}-1}$ of boundary coefficients is given by the expression $[B_{\rm p}/(1+B_{\rm p})]^{B_{\rm p}+1}$. When $B_{\rm p} \to \infty$ $c_{K_{\rm ent}-1}/c_{R_{\rm s}-1} \to 1/e$.

To compare the values of entanglement dimensionality and the participation ratio for general twin beams we have to eliminate the effect of different boundaries set by different definitions, as revealed by considering the pure states. Using the formulas derived for noiseless twin beams, we introduce the modified entanglement dimensionality $\tilde{K}_{\rm ent}$ as follows:

$$\tilde{K}_{\rm ent} = \frac{2B_{\rm p}+1}{2B_{\rm p}+1+2\sqrt{B_{\rm p}^2 + B_{\rm p}}} K_{\rm ent}. \qquad (31)$$

Definition (31) of the modified entanglement dimensionality $\tilde{K}_{\rm ent}$ guarantees that the values of modified entanglement dimensionality $\tilde{K}_{\rm ent}$ and participation ratios $R_{\rm s}$ and $R_{\rm i}$ of noiseless twin beams are equal.

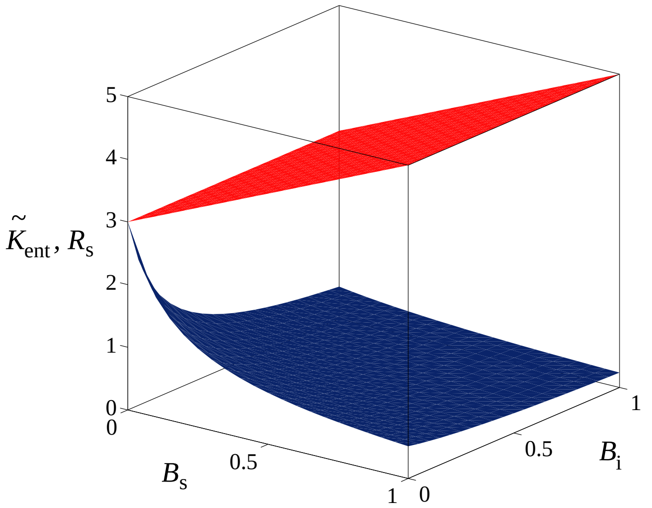

FIG. 8: (Color online) Modified entanglement dimensionality $\tilde{K}_{\rm ent}$ given in Eq. (31) [lower, dark-gray (blue) area] and signal-field participation ratio $R_{\rm s}$ given in Eq. (27) [upper, gray (red) area] as they depend on the mean noise photon numbers $B_{\rm s}$ and $B_{\rm i}$ assuming the mean photon-pair number $B_{\rm p} = 1$.

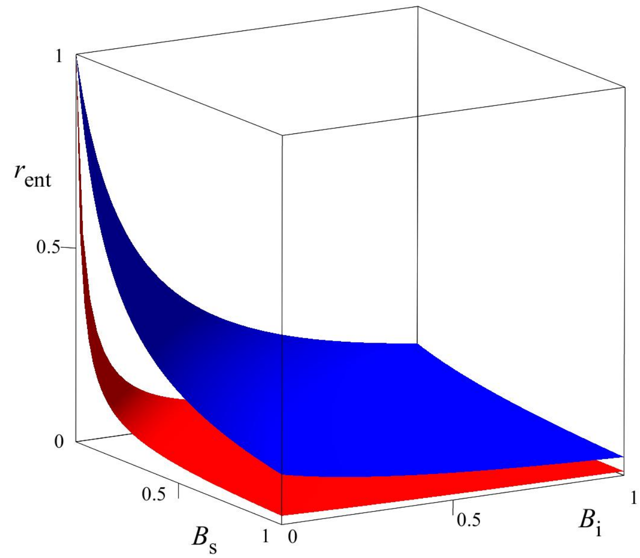

FIG. 9: (Color online) Coefficient $r_{\rm ent}$ given in Eq. (32) versus the mean noise photon numbers $B_{\rm s}$ and $B_{\rm i}$ for the mean photon-pair number $B_{\rm p}$ equal to 1 [upper, dark-gray (blue) area], and 10 [lower, gray (red) area].

The values of the modified dimensionality $\tilde{K}_{\rm ent}$ of entanglement and the signal-field participation ratio $R_{\rm s}$ are compared in Fig. 8 for the mean photon-pair number $B_{\rm p} = 1$. Whereas the values of the modified entanglement dimensionality $\tilde{K}_{\rm ent}$ decrease with increasing values of the mean noise photon numbers $B_{\rm s}$ and $B_{\rm i}$, the values of the signal-field participation ratio $R_{\rm s}$ increase with increasing values of the mean signal-field noise photon number $B_{\rm s}$. We note that the values of the signal-field participation ratio $R_{\rm s}$ are greater than those of the modified entanglement dimensionality $\tilde{K}_{\rm ent}$ even for $B_{\rm s} = 0$, as the presence of noise in the idler field ($B_{\rm i} > 0$) degrades entanglement.

The relative contribution of the degrees of freedom used for describing entanglement in a twin beam is an important characteristic. This contribution can be quantified via the coefficient $r_{\rm ent}$ defined as follows:

$$r_{\rm ent} = \frac{2\tilde{K}_{\rm ent}}{R_{\rm s} + R_{\rm i}}. \tag{32}$$

As shown in Fig. 9, the greater the values of the mean noise photon numbers $B_{\rm s}$ and $B_{\rm i}$, the smaller the values of the coefficient $r_{\rm ent}$. The comparison of surfaces of the coefficient $r_{\rm ent}$ drawn for the mean photon-pair numbers $B_{\rm p} = 1$ and $B_{\rm p} = 10$ in Fig. 9 reveals seemingly paradoxical behavior. The values of the coefficient $r_{\rm ent}$ decrease with increasing values of the mean photon-pair number $B_{\rm p}$. This behavior, however, naturally originates in fragility of entanglement with respect to the noise. More intense twin beams (with greater values of $B_{\rm p}$) are less resistant to a given amount of noise compared to low-intensity twin beams. This is explained by the larger dimensions of the effectively populated Hilbert spaces of more intense twin beams and, thus, the more complex structures of their entanglement. As a consequence, relatively higher numbers of degrees of freedom serving to describe entanglement in more intense noiseless twin beams are "released" by the noise and enlarge the noise parts of twin beams.

Alternatively to the participation ratio R, we may apply the von Neumann entropy S of a reduced statistical operator. Taking into account the diagonal form of the signal-field reduced statistical operator $\hat{\rho}_{\rm s}$ with the elements written in Eq. (26), the signal-field entropy $S_{\rm s}$ is in general determined along the formula

$$S_{\rm s} = -{\rm Tr}(\hat{\rho}_{\rm s} \ln \hat{\rho}_{\rm s}) = -\sum_j \rho_{{\rm s},jj} \ln(\rho_{{\rm s},jj}). \tag{33}$$

Considering the specific form of matrix elements $\rho_{{\rm s},jj}$ given in Eq. (26), the formula for entropy $S_{\rm s}$ attains the form

$$\begin{aligned} S_{\rm s} &= (1 + B_{\rm p} + B_{\rm s}) \ln(1 + B_{\rm p} + B_{\rm s}) \\ &\quad - (B_{\rm p} + B_{\rm s}) \ln(B_{\rm p} + B_{\rm s}); \end{aligned} \tag{34}$$

ln stands for natural logarithm. Combining Eqs. (25) and (34), the entropy $S_{\rm s}$ is revealed as an increasing function of the participation ratio $R_{\rm s}$:

$$S_{\rm s} = \frac{1}{2} \left[(R_{\rm s} + 1) \ln(R_{\rm s} + 1) - (R_{\rm s} - 1) \ln(R_{\rm s} - 1) \right] - 1. \tag{35}$$

Analogous formulas for the idler-field entropy $S_{\rm i}$ can easily be derived. The general dependence of entropy $S_{\rm s}$ on

the participation ratio R_{s} is plotted in Fig. 10. We would like to note that the entropy S serves as a good measure of the entanglement for pure states.

VI. TWIN BEAM COMPOSED OF M MODES

In real experiments, twin beams are rarely composed of only one paired spatiotemporal mode [30, 34]. We note that a twin beam composed of one paired mode represents an ideal field from the experimental point of view [67]. For this reason, we consider a multimode twin beam containing M independent identical single-mode twin beams. Its statistical operator $\hat{\rho}_M$ is given as $\hat{\rho}_M = \otimes_M \hat{\rho}$ using the statistical operator $\hat{\rho}$ written in Eq. (8). There are four parameters characterizing the twin beam: number M of modes, mean photon-pair number B_{p}, mean signal-field noise photon number B_{s}, and mean idler-field noise photon number B_{i}. We note that such an M-mode twin beam represents a good approximation of a real twin beam when all spatiotemporal modes participating in the nonlinear interaction are detected.

The considered physical quantities behave differently with respect to the number M of modes. It has been shown in Refs. [59] and [34] that the nonclassical depth τ does not depend on the number M of modes. On the other hand, the multimode negativity N_M, $N_M = (1+2N)^M$, as well as the participation ratios $R_{M,a}$, $R_{M,a} = R_a^M$ for $a = s, i$, are multiplicative. We note that the form of the multimode negativity originates in the multiplicative property of the trace norm and its relation to the negativity expressed in Eq. (11) [41]. In fact, the multimode negativity N_M coincides with the entanglement dimensionality K_{ent} defined in Eq. (23) for a single-mode twin beam. The multimode entropies $S_{M,a}$, $a = \mathrm{s}, \mathrm{i}$, are then additive. To reveal similar relations among the studied quantities as has been done for single-mode twin beams, we have to define suitable quantities derived from those considered above. Defining the logarithmic negativity $N_M^{\log} \equiv \ln(N_M)$ and the logarithmic participation ratios $R_{M,a}^{\log} \equiv \ln(R_{M,a})$, $a = \mathrm{s}, \mathrm{i}$, we replace the multiplicative quantities with the additive ones. Introducing the logarithmic negativity $\mathcal{N}$, logarithmic participation ratios $\mathcal{R}_a^{\log}$, and entropies $\mathcal{S}_a$ related per one mode,

$$
\begin{aligned}
\mathcal{N} &= \frac{N_M^{\log}}{M} = \ln(1+2N), \\
\mathcal{R}_a &= \frac{R_{M,a}^{\log}}{M} = \ln(R_a), \\
\mathcal{S}_a &= \frac{S_{M,a}}{M} = S_a,
\end{aligned} \tag{36}
$$

with $a = \mathrm{s}, \mathrm{i}$, we reveal the suitable quantities. The quantities defined in Eq. (36) together with the nonclassical depth τ behave qualitatively in the same way as those defined for single-mode twin beams discussed above. Especially, the logarithmic negativity $\mathcal{N}$ per mode is an increasing function of the nonclassical depth τ. Also, the entropy $\mathcal{S}_a$ per mode is an increasing function of the logarithmic participation ratio $\mathcal{R}_a$ per mode, $a = \mathrm{s}, \mathrm{i}$.

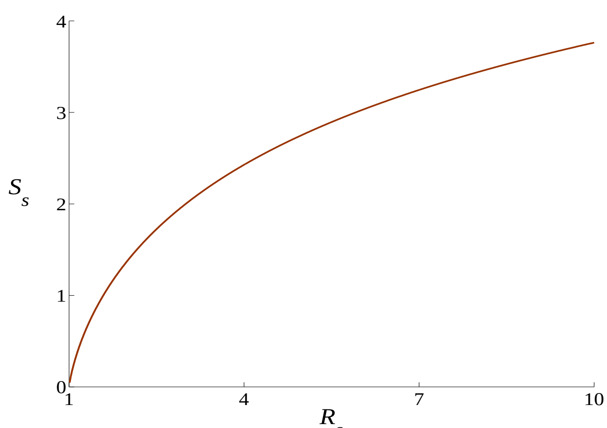

FIG. 10: (Color online) von Neumann entropy S_{s} as a function of the participation ratio R_{s} according to Eq. (35).

VII. EXPERIMENTAL MULTIMODE TWIN BEAMS

Real experimental multimode twin beams have a more complex structure than that discussed in Sec. VI [30, 32, 34]. The reason is that the spatiotemporal modes of twin beams are shared by the signal and idler fields and so they can be broken before or during the detection owing to spectral and/or spatial filtering. As a consequence, real multimode twin beams are composed of three components [8, 34]. A paired component describes photons embedded in spatiospectral modes detected by both signal- and idler-field detectors. A noise signal (idler) component then describes photons occurring in signal (idler) spatiotemporal modes that originate in filtering of the idler (signal) field. If we assume for simplicity that the paired component is ideal, i.e., without noise, we need six parameters to describe a real twin beam. Each component is characterized by the number M of modes and mean photon-pair (or noise photon) number B. The statistical operator $\hat{\rho}_E$ of the experimental twin beam can be expressed as

$$
\hat{\rho}_E = \bigotimes_{M_{\mathrm{p}}} \hat{\rho}_{\mathrm{p}} \bigotimes_{M_{\mathrm{s}}} \hat{\rho}_{n,s} \bigotimes_{M_{\mathrm{i}}} \hat{\rho}_{n,\mathrm{i}} \tag{37}
$$

using single-mode statistical operators $\hat{\rho}_{\mathrm{p}}$, $\hat{\rho}_{n,s}$, and $\hat{\rho}_{n,i}$ of the photon-pair, noise signal, and noise idler components. In Eq. (37), M_{p}, M_{s}, and M_{i} give the numbers of equally populated modes with the mean numbers B_{p}, B_{s}, and B_{i} of photon pairs per mode, respectively.

Entanglement in the experimental twin beam is created only by its paired component and as such it can be

quantified by the logarithmic negativity $N^{\log}_{M_{\rm p}}$ introduced in Sec. VI. The noise components do not contribute to entanglement on one side, and they do not degrade entanglement on the other side. This is qualitatively different from the case of multimode twin beams discussed in Sec. VI and containing noise in paired spatiotemporal modes.

Nonclassicality can be quantified by a multimode generalization of nonclassical depth τ_E introduced in Ref. [45] for a single-mode field. In a multimode twin beam, we may first determine the standard nonclassical depths τ_n for each single-mode field, included either in the paired part of the twin beam or in the noisy signal and idler parts of the twin beam. Then we can take either $\max_n(\tau_n)$ or $\sum_n \tau_n$ to quantify the multimode nonclassical depth τ_E. In the first case, the nonclassical depth τ_E of the experimental multimode twin beam is just given by the nonclassical depth τ of a paired mode. The second case is physically more interesting, as the value of τ_E is linearly proportional to the minimum amount of additional noise needed to conceal nonclassicality of the multimode state. In this case, we have, for the experimental multimode twin beams,

$$\tau_E = M_{\rm p}\tau. \tag{38}$$

Using the logarithmic negativity $N^{\log}_{M_p}$ defined in Sec. VI and the nonclassical depth τ_E, one-to-one correspondence between the entanglement and the nonclassicality is obtained also for M-mode twin beams.

On the other hand, the concept of weak nonclassicality [25, 68, 69] is also useful for the experimental multimode twin beams considered to be composed of one effective paired (macro)mode. The joint quasidistribution P_W of the integrated intensities W_s and W_i of the signal and idler fields, respectively, describes the properties of this effective paired mode [11]. As no information about the phase is encoded in this simplified effective description, we may only determine the nonclassical intensity depth τ_W quantifying nonclassicality, which demonstrates itself by negative values of the marginal quasidistribution of integrated intensities. We have to emphasize that the nonclassical intensity depth τ_W is only a nonclassicality witness or parameter, which reveals nonclassicality solely in photon-number statistics. Contrary to this, the nonclassical depth τ is a genuine and commonly used nonclassicality measure. We note that te standard nonclassicality quantified by τ reveals both strongly and weakly nonclassical states [68, 69]. From this point of view τ is a *strong* tool or criterion. On the other hand, τ_W detects only strongly nonclassical states; i.e., it is a *weak* tool.

The nonclassical intensity depth τ_W has been determined for the experimental multimode twin beams in Ref. [34],

$$\tau_W = \sqrt{\beta^2-\gamma}-\beta, \tag{39}$$

where

$$\beta = \frac{M_{\rm s}B_{\rm s}+M_{\rm i}B_{\rm i}+2M_{\rm p}B_{\rm p}}{M_{\rm s}+M_{\rm i}+2M_{\rm p}},$$

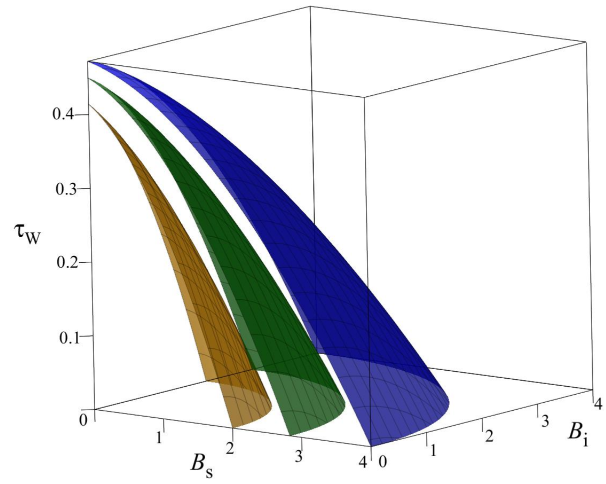

FIG. 11: (Color online) Nonclassical intensity depth τ_W as a function of the mean noise photon numbers $B_{\rm s}$ and $B_{\rm i}$ for the mean photon-pair number $B_{\rm p}$ equal to 2 [bottom, light-gray (yellow) area], 4 [gray (green) area], and 8 [top, dark-gray (blue) area], assuming $M_{\rm p}=M_{\rm s}=M_{\rm i}=1$. The greater the value of $B_{\rm p}$, the greater the value of τ_W.

$$\gamma = \frac{M_{\rm s}B_{\rm s}^2+M_{\rm i}B_{\rm i}^2-2M_{\rm p}B_{\rm p}}{M_{\rm s}+M_{\rm i}+2M_{\rm p}}. \tag{40}$$

The analysis of Eq. (39) shows that the experimental multimode twin beam is strongly nonclassical ($\tau_W>0$) provided that

$$M_{\rm s}B_{\rm s}^2+M_{\rm i}B_{\rm i}^2 < 2M_{\rm p}B_{\rm p}. \tag{41}$$

Inequality (41) means that the multimode strong nonclassicality of the twin beam is lost if the noise is sufficiently strong. For example, if $M_{\rm p}=M_{\rm s}=M_{\rm i}$, strongly nonclassical multimode twin beams are observed for $B_{\rm s}^2+B_{\rm i}^2<2B_{\rm p}$ (see Fig. 11). This behavior is similar to that discussed in Sec. IV, though the boundary given by $\tau_W=0$ is quantitatively different (compare Figs. 6 and 11). We also have here that the greater the value of the mean photon-pair number $B_{\rm p}$, the greater the value of the nonclassical intensity depth τ_W. Also, the greater the values of mean noise photon numbers $B_{\rm s}$ and $B_{\rm i}$, the smaller the value of the nonclassical intensity depth τ_W.

Similarly as in Sec. VI, the logarithmic participation ratio $R^{\log}$ can be defined for each component of the twin beam to quantify its dimensionality. The logarithmic participation ratio $R^{\log}$ of the whole twin beam is then naturally given as the sum of the logarithmic participation ratio $R^{\log}_{M_{\rm p},{\rm p}}$ of the paired component and the logarithmic participation ratio $R^{\log}_{M_{\rm s},{\rm s}}+R^{\log}_{M_{\rm i},{\rm i}}$ of the noise signal and idler components. We note that Eq. (25) is appropriate for determining the participation ratio of both the single-mode noise signal (or idler) field and the single-mode paired field. Alternatively we may consider

entropies of the components instead of participation ratios. Entropies of the single-mode noise fields are given by Eq. (33). Equation (33) is applicable also for determination of the entropy of entanglement of a single-mode paired field in a pure state for which $\hat{\rho}_{\mathrm{s},jj} \leftarrow c_j^2$. As a consequence, the entropies $S_{M_a,a}$ for $a = \mathrm{p,s,i}$, of each component are increasing functions of the corresponding participation ratios $R_{M_a,a}$. In single-mode cases, these functions are determined by Eq. (35), plotted in Fig. 10. Similarly to the overall logarithmic participation ratio $R^{\log}$, the overall entropy S can be naturally split into its entangled part $S_{M_\mathrm{p},\mathrm{p}}$ and noisy part $S_{M_\mathrm{s},\mathrm{s}} + S_{M_\mathrm{i},\mathrm{i}}$, originating in the noise signal and idler components.

Finally, we briefly address the issue of the experimental determination of the quantities discussed above. As these quantities characterize the "internal" structure of a twin beam, only their indirect determination is possible. It is based upon the measurement of the joint signal-idler photocount histogram using photon-number-resolving detectors. Knowing these detector parameters [33], reconstruction of the joint signal-idler photon-number distribution [8, 34] provides the applied mean photon(-pair) numbers B and numbers M of modes. The above-derived formulas then give the discussed quantities.

VIII. CONCLUSIONS

The entanglement and nonclassicality of a single-mode noisy twin beam have been quantified using the negativity and the nonclassical depth, respectively. Universal mapping between the nonclassical depth and the negativity has been revealed for noisy twin beams. The mapping reflects the fact that nonclassicality of a twin beam is caused by the entanglement of its two parts originating in pairing of photons. Limitations to the amount of noise have been found to preserve entanglement together with nonclassicality. the degrees of freedom of a twin beam quantified by the signal- and idler-field participation numbers have been divided into those needed to describe entanglement and the remaining ones forming the noisy signal and idler parts of the twin beam. The entanglement dimensionality derived from the negativity has been applied here. Entropy as an increasing function of the participation number has been discussed. Properties of multimode twin beams have been analyzed using appropriate quantities related per one mode. Also, experimental multimode twin beams containing additional noise in independent spatiotemporal modes have been investigated from the point of view of their entanglement and multimode nonclassicality including weak nonclassicality and dimensionality.

Acknowledgments

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